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Exploring the Fourier Series by Drawing Figures with Circles

Introduction

One of the most famous works that Joseph Fourier was his discovery of the Fourier series and transformation, which expresses the idea that that any function could be represented with a combination of its constituent frequencies. Although Fourier analysis was discovered in the early 19th century, early ideas of decomposing functions to simple oscillating functions can be dated back to 3rd century B.C., when ancient astronomers proposed mathematical models based on the concepts of deferent and epicycles to describe planetary motions (Sargent, 1917). In addition, Fourier analysis, the study of decomposing periodic waveforms into their corresponding composite trigonometric functions, has a wide array of applications in image and music compression, optics, cryptography, signal processing, and many more areas that are highly pertinent to our daily life (Niu, 2006). Having learned about the ancient astronomical models in physics and the importance of Fourier analysis, I decided to combine, understand, and explore these concepts in a creative way by drawing images with combinations of circles. Since there has been relatively less research done on drawing images with combinations of circles using Fourier series, it was really exciting for me to investigate this topic.

Background

In Ptolemy's (an ancient Greek astronomer) model, each planet moves in an epicycle (a small circle) centered on the path of a deferent (a larger circle), as shown in Fig. 1; both of these terms will be used when describing the circles for clarification (The Ptolemaic Model, n.d.).

Some of the common image compression techniques that we currently have are JPEG and PNG files. Although both are common for digital images, JPEG images use the discrete cosine transform, a variant of the Fourier series, to convert an image into a series of frequencies, whereas PNG images store the value of all pixels in a 2-dimensional array (Mathis, 2015). The distinctions in these methods of compression result in a difference in the quality and size of the compressed images, with the PNG images normally having a larger size but better quality (Mathis, 2015). Some properties of compression through Fourier series will be explored in more depth in further sections.

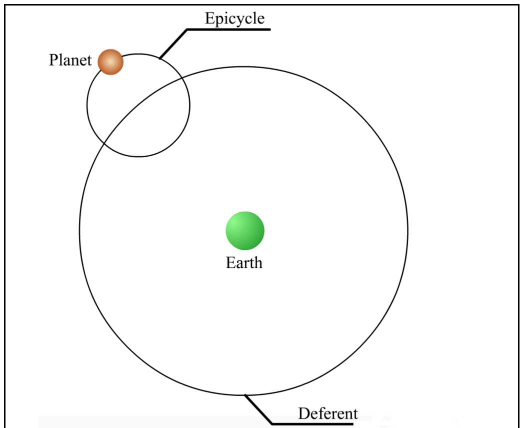

Fig. 1 A Diagram of an Epicycle and Deferent (The Ptolemaic Model, n.d.)

Orthogonality

The orthogonality relation states that the integral of the product of two functions $u(t)$ and $v(t)$, where $u(t)$ and $v(t)$ represent either a cosine or sine function, integrated over a period is equal to 0, as long as $u(t) \neq v(t)$ (Orthogonality Relations, 2011). The proof is not shown as it is just a straightforward calculation.

$$\int_{-\pi}^{\pi} u(t)v(t)dt = 0$$

The orthogonality could be broken down into a few cases as shown below, where $m \in \mathrm{N}$ and $n \in \mathrm{N}$.

$$\int_{-\pi}^{\pi} \sin(mt)\,dt = \int_{-\pi}^{\pi} \cos(mt)dt = \int_{-\pi}^{\pi} \sin(mt)\cos(nt)\,dt = 0$$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} \pi, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$$

Theorem

The Fourier series starts out with the idea that all periodic functions can be made up of sine and cosine functions, which was then defined as a periodic function made up of an infinite series (or linear combination) of weighted harmonic sinusoids (Cheever, n.d.).

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Given $f(t)$ with a period of 2π, where k and n represent arbitrary integers and a_n and b_n represent the "weight" of each sinusoidal function, the Fourier series of $f(t)$ can be expressed as the following equation (Cheever, n.d.).

$$f(t) = c_0 + \cdots + a_k\cos(kt) + \cdots + a_n\cos(nt) + \cdots = c_0 + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$$

This equation is the Fourier synthesis equation, as it shows how an equation is synthesized by adding up sine and cosine functions. Since the functions must be harmonics of one another, k and n must be integers in order to complete an integer number of oscillations in a period. a_n, b_n, and c_0 are constants that are called the Fourier coefficients, and they will be further discussed in the following section.

Finding the Fourier Coefficients a_n and b_n and the Constant Term c_n

Using the orthogonality relation, a_n could be found by multiplying $f(t)$ with $\cos(kt)$ and taking the integral of $f(t)$ over a period of 2π (from $-\pi$ to π) (Cheever, n.d.). By convention, the constant of integration is not included in the calculations since the constant term for the Fourier series (c_0) is calculated separately.

$$f(t)\cos(nt) = c_0\cos(nt) \ldots + a_k\cos(\mathrm{kt})\cos(nt) + \cdots + a_n\cos^2(nt) + \cdots$$

Taking the integral of $f(t)\cos(nt)$:

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = c_0\int_{-\pi}^{\pi}\cos(nt)\,dt + \cdots + \int_{-\pi}^{\pi} a_k\cos(kt)\cos(nt)\,dt + \cdots + \int_{-\pi}^{\pi} a_n\cos^2(nt)\,dt$$

$$= c_0\cdot 0 + \cdots + a_k\cdot 0 + a_n\cdot\left(\frac{t}{2} + \frac{\sin(2nt)}{4n}\right)\Bigg|_{-\pi}^{\pi} = a_n\cdot\{(\frac{\pi}{2} + \frac{\sin(2n\pi)}{4n}) - (\frac{-\pi}{2} + \frac{\sin(-2n\pi)}{4n})\} = a_n\cdot\pi$$

b_n can also be found in a similar manner by multiplying $f(t)$ with $\sin(nt)$ and taking the integral over its period. Through inspection of the Fourier synthesis equation, it can be found out that c_0 can be determined by multiplying the function $f(t)$ with $\cos(0\cdot t) = 1$ and integrating the function over its period.

$$\int_{-\pi}^{\pi} f(t)dt = c_0\int_{-\pi}^{\pi} dt + \cdots + \int_{-\pi}^{\pi} a_k\cos(kt)\,dt + \cdots + \int_{-\pi}^{\pi} a_n\cos(nt)\,dt = c_0\cdot x|_{-\pi}^{\pi} + \cdots + a_k\cdot 0 + a_n\cdot 0 = c_0\cdot 2\pi$$

Since $c_0 = \frac{1}{T}\int_{-\pi}^{\pi} f(t)dt$, where T represents the period of the function, c_0 also represents the average value of the function. Substituting 0 for n in a_n, where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(0 * dt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$, c_0 can also be expressed as $\frac{a_0}{2}$. This is verified since an odd function, which has a Fourier series that is composed of only sine functions, would have an average of 0, whereas an even function may have a non-zero average.

The following relation, which is called the Fourier analysis equation, can be established:

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$	$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$	$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t$

Generalizing to Different Periods

The orthogonality relation could be expanded to any arbitrary period, as $f(x)$ with period 2π and domain $x \epsilon [-\pi, \pi]$ could be viewed as a special case of a function with period 2L, where L ϵ R.

Let the period of a periodic function be $2L$, where $\cos(\frac{n\pi t}{L})$ and $\sin(\frac{n\pi t}{L})$ represent $\cos(nt)$ and $\sin(nt)$ when $2L = 2\pi$. Thus, the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$ with a period of 2π can then be generalized as $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin(\frac{n\pi t}{L})$ with a period of $2L$ for L ϵ R. Since the orthogonality relation for the sine and cosine functions holds for any function integrated over its period, the Fourier analysis equation can then be expressed in terms of the following relation.

$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$	$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$	$c_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\mathrm{d}t$

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Euler's Equation and the Exponential Form of the Fourier series

Euler's formula $e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$ simplifies the equation $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$ by converting the equation to its exponential form. A complex number in the form of $z = a + bi$ can be viewed as a cartesian coordinate at point (a, b) in the complex plane, which could be converted into the polar coordinates (r, θ) in the form $a = r\cos(\theta)$ and $b = r\sin(\theta)$, where r represents the radius. The complex number z can then be expressed in its polar form as $z = r(\cos(\theta) + i \cdot \sin(\theta)) = rcis(\theta) = re^{i\theta}$, where $r = |z| = \sqrt{a^2 + b^2}$.

In order for the circle to rotate with a change in time, t is substituted with the angle θ, which gives the equation $z = re^{it}$. As θ moves from 0 to 2π, z traces out a circle with radius r. The angular frequency of the rotation can also be described in terms of by ω, expressed in terms of radians per second, where $\omega = \frac{d\theta}{dt} = \frac{2\pi}{T}$ (Gupta, 2019). The formula $re^{i\omega t}$ then represents a circle of radius r that rotates at an angular frequency of ω. Similarly, by and plugging 2L into T, ω could be substituted with $\frac{\pi}{L}$ and $f(t)$ can also be simplified to $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$.

It can be known that $cis(-\omega t) = \cos(-\omega t) + i \cdot \sin(-\omega t) = \cos(\omega t) - i \cdot \sin(\omega t)$ since cosine is an even function and sine is an odd function. Using De Moivre's theorem, $cis(-\omega t) = cis(\omega t)^{-1}$, and since $cis(\omega t) = e^{i\omega t}$, $cis(-\omega t) = e^{-i\omega t}$ (Demenet, Nirjhor, & Khan, n.d.). Using this notation, the following relation can be established.

$$\begin{cases} \dfrac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos(\omega t) \\ \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i} = \sin(\omega t) \end{cases}$$

This allows the Fourier synthesis equation to be converted to its exponential form $f(x) = \sum_{-\infty}^{\infty} c_n e^{in\omega t}$. Let the constant term c_0 represent the average value of a function. Since $e^{in\omega t}$ rotates at a frequency of ω radians per second, it completes a n cycles in a second and has a frequency of n Hz. Therefore, $c_{-n}e^{-in\omega t} + c_n e^{in\omega t}$ represents the portion of $f(t)$ that makes a n oscillations in a second, which is also equal to $a_n \cos(n\omega t) + b_n \sin(n\omega t)$ in the synthesis equation of the Fourier series of $f(t)$ (Cheever, n.d.).

By equating the equations of $a_n \cos(n\omega t) + b_n \sin(n\omega t)$ and $c_{-n}e^{-in\omega t} + c_n e^{in\omega t}$, the Fourier coefficients c_n can be calculated and expressed in terms of a_n and b_n. Using the relation of expressing cosine and sine functions in terms of the addition and subtraction of Euler's formulas as defined above, the equation can be written as follows.

$$a_n\left(\frac{e^{in\omega t} + e^{-in\omega t}}{2}\right) + b_n\left(\frac{e^{in\omega t} + e^{-in\omega t}}{2i}\right) = \frac{a_n}{2}e^{in\omega t} + \frac{b_n}{2i}e^{in\omega t} + \frac{a_n}{2}e^{-in\omega t} - \frac{b_n}{2i}e^{in\omega t} = c_{-n}e^{-in\omega t} + c_n e^{in\omega t}$$

Since $\frac{b_n}{2i}$ can also be written as $\frac{b_n}{2} \cdot i^{-1 \bmod 4} = \frac{b_n}{2} \cdot i^3 = \frac{b_n}{2} \cdot i^2 \cdot i = -\frac{b_n}{2} \cdot i$, by grouping the equations in terms of the terms $e^{in\omega t}$ and $e^{-in\omega t}$, the following expression could be established.

$$\begin{cases} c_n = \frac{a_n}{2} - \frac{b_n}{2}i \\ c_{-n} = \frac{a_n}{2} + \frac{b_n}{2}i \end{cases} \text{for } n \neq 0 \text{ with } c_n^* = c_{-n}$$

Assuming that $f(t)$ is a real function, the imaginary parts of c_n and c_{-n} must cancel out to become 0. As a result, c_n and c_{-n} are complex conjugates, which is shown in the expression established above. However, a real function expressed in terms of time t is not sufficient to plot a 2-dimensional closed-loop figure on a plane as it fails the vertical line test (unless it is a parametric equation), which implies that the function should be plotted on a complex plane in order for the function to be plotted without using parametric equations. When graphed on a complex plane, a real function also traces out a line in the real number axis, meaning that the functions that is used to plot the figure should not be a real function. This also signifies that c_n and c_{-n} should not be complex conjugates or else the imaginary numbers would cancel each other out.

The exponential form of the Fourier series also provides a convenient way to apply a phase shift to a term in the Fourier series. By multiplying a term $c_n e^{in\omega t}$ with e^{ϕ_n}, the term then becomes $c_n e^{in\omega t + \phi_n}$, which could be viewed as a function with a phase shift by ϕ_n radians.

Revisiting the idea of viewing complex exponential functions as rotations in the complex plane takes us a step further to plot a two-dimensional figure with circles. The sum of the terms in the Fourier series in the exponential form can be represented as having an epicycle (a term in the series) rotate about another deferent (another term in the series). It should be noted that the commutative representation of the sum of the Fourier series is not unique since the terms of the Fourier series can be organized in any order following the commutative property of addition.

Example of a Calculation for the Fourier Series

Let us take these concepts and apply it to a real-life example. Below (Fig. 2) is an image of a maple leaf, which is a suitable image for Fourier epicycles to trace out as the figure has a clear outline. Before plotting the whole leaf, let us zoom into a corner of the figure (as shown in Fig. 3) and derive the trigonometric Fourier series of the curve.

Fig. 2 Figure of a Maple Leaf

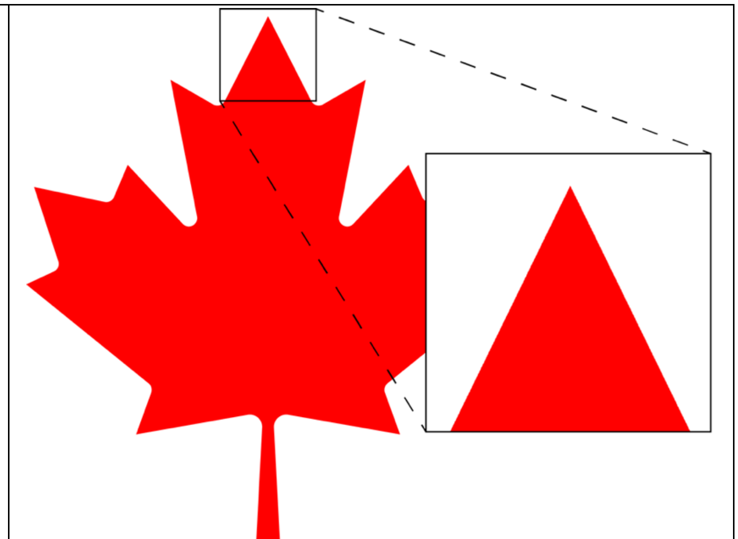

Fig. 3 Section of a Maple Leaf

(National flag of Canada (outdoor use) .: P29-098-001-2018E-PDF, 2013)

The zoomed in portion of the figure is a triangle with a height-to-width ratio of roughly 1:1. The outline of the triangle is then plotted onto a plane with an arbitrary unit axis, as shown in Fig. 4.

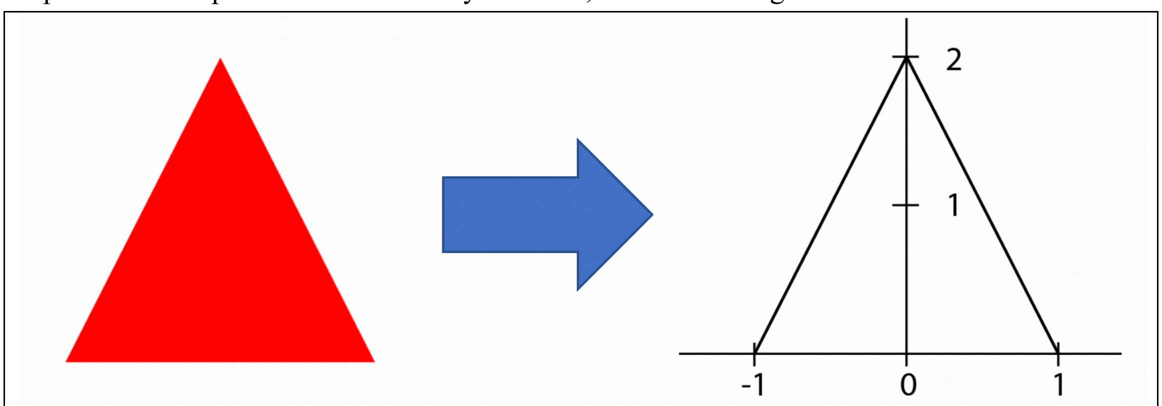

Fig. 4 A Plot of the Maple Leaf Section on a Plane

This curve represents a piecewise function with the formula $f(t) = \begin{cases} 2t+2, -1 < x \leq 0 \\ -2t+2, 0 < x \leq 1 \end{cases}$. The Fourier series would trace out a similar function that is repeated periodically and infinitely along the x-axis. Since the function $f(t)$ is an even function, the Fourier series is a Fourier cosine series, in which the terms b_n are all equal to 0, and a proof will be shown in the next section. $f(t)$ is piecewise continuous as both $f(t) = 2t + 2$ and $f(t) = -2t + 2$ are continuous, and the Fourier coefficients can be calculated because the definite integrals in a Fourier series for a piecewise continuous function will always converge (Tseng, 2012).

The following steps shown the calculation of the Fourier coefficients.

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Since the period equals to $2 = 2L$ (from -1 to 1), L equals to 1 for the following calculations.

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(t)\,dt = \frac{1}{2}\left(\int_{-1}^{0} 2t + 2\,dt + \int_{0}^{1} -2t + 2\,dt\right) = \int_{-1}^{0} t + 1\,dt + \int_{0}^{1} -t + 1\,dt = (\frac{t^2}{2}+t)|_{-1}^{0} + (-\frac{t^2}{2} + t)|_{0}^{1}$$

$$= \left(0 - \left(\frac{1}{2} - 1\right)\right) + \left(\left(-\frac{1}{2} + 1\right) - 0\right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos(n\pi t)\,dt = \int_{-1}^{0} (2t+2)\cdot\cos(n\pi t)\,dt + \int_{0}^{1} (-2t+2)\cdot\cos(\mathrm{n\pi t})\,\mathrm{dt}$$

$$= \int_{-1}^{0} 2t\cdot\cos(n\pi t)\,dt - \int_{0}^{1} 2t\cdot\cos(n\pi t)\,dt + 2\int_{-1}^{1}\cos(n\pi t)\,dt$$

Breaking down the equation,

$$2\int_{-1}^{1}\cos(n\pi t)\,dt = 2\cdot\frac{1}{\pi n}\sin(n\pi t)\,|_{-1}^{1} = 2\cdot\left(\frac{\sin(\pi n)}{\pi n} - \frac{\sin(-\pi n)}{\pi n}\right) = 4\cdot\left(\frac{\sin(\pi n)}{\pi n}\right) = 4\cdot 0 = 0$$

Using integration by parts,

$$\int 2t\cdot\cos(n\pi t)\,dt = 2t\cdot\frac{1}{n\pi}\sin(n\pi t) - \int\frac{1}{n\pi}\sin(n\pi t)\cdot 2\,dt = \frac{2t}{n\pi}\sin(n\pi t) - \frac{-2}{n^2\pi^2}\cos(n\pi t)$$

$$= \frac{2}{n^2\pi^2}(\pi nt\cdot\sin(n\pi t) + \cos\,(n\pi t))$$

a_n can then be calculated by substituting the integral with a definite integral as shown below.

$$a_n = \int_{-1}^{1} 2t\cdot\cos(n\pi t)\,dt = \frac{2}{n^2\pi^2}(\pi nt\cdot\sin(n\pi t) + \cos\,(n\pi t))|_{-1}^{0} - \frac{2}{n^2\pi^2}(\pi nt\cdot\sin(n\pi t) + \cos(n\pi t))|_{0}^{1} + 0$$

$$= \frac{2}{n^2\pi^2}\{[(0+1) - (\pi n\cdot\sin(\pi n) + \cos(-n\pi))] - [(\pi n\cdot\sin(\pi n) + \cos(n\pi)) - (0+1)]\}$$

$$= \frac{2}{n^2\pi^2}\cdot\big((1 - \pi n\cdot\sin(\pi n) - \cos(n\pi)) - (\pi n\cdot\sin(\pi n) + \cos(n\pi) - 1)\big)$$

$$= \frac{2}{n^2\pi^2}\{[1 - (-1)^n] - [(-1)^n - 1]\} = \frac{4}{n^2\pi^2} - \frac{4(-1)^n}{n^2\pi^2} = \frac{4 - 4(-1)^n}{n^2\pi^2}$$

Verifying that $c_0 = \frac{a_0}{2}$ using the equation of a_n:

$$a_0 = \int_{-1}^{0} 2t\cdot\cos(0)\,dt - \int_{0}^{1} 2t\cdot\cos(0)\,dt + 2\int_{-1}^{1}\cos(0)\,dt = t^2|_{-1}^{0} - t^2|_{0}^{1} + 2x|_{-1}^{1} = -1 - 1 + 2 - (-2) = 2$$

$$\therefore \frac{a_0}{2} = \frac{2}{2} = 1 = c_0$$

Proof that the Fourier Series is a Cosine Series ($b_n = 0$)

Assuming that $b_n \neq 0$ for a Fourier series of $f(t)$, where $f(t)$ is an even function in the real-number domain,

$$\sum_{n=1}^{\infty} f(t) = \sum_{n=1}^{\infty} f(-t) = \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right) = \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{-n\pi t}{L}\right)$$

This creates a contradiction as $b_n\sin\left(\frac{n\pi t}{L}\right)$ should equal $b_n\sin\left(\frac{-n\pi t}{L}\right) = -b_n\sin\left(\frac{n\pi t}{L}\right)$ unless $nt = L$ so that $\sin\left(\frac{n\pi t}{L}\right) = 0$. Since n changes as the Fourier coefficients are summed up and t changes with the function, the only way for the Fourier series to satisfy all terms for any given t is for $b_n = 0$.

A quick calculation helps us to verify this statement.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin(n\pi t)\,dt = \int_{-1}^{0} (2t+2)\cdot\sin(n\pi t)\,dt + \int_{0}^{1} (-2t+2)\cdot\sin(\mathrm{n\pi t})\,\mathrm{dt}$$

$$= \int_{-1}^{0} 2t\cdot\sin(n\pi t)\,dt - \int_{0}^{1} 2t\cdot\sin(n\pi t)\,dt + 2\int_{-1}^{1}\sin(n\pi t)\,dt$$

Breaking down the equation,

$$2\int_{-1}^{1}\sin(n\pi t)\,dt = -2\cdot\frac{1}{\pi n}\cos(n\pi t)\,|_{-1}^{1} = -2\cdot\left(\frac{\cos(\pi n)}{\pi n}-\frac{\cos(-\pi n)}{\pi n}\right) = 2\cdot 0 = 0$$

Using integration by parts,

$$\int 2t\cdot\sin(n\pi t)\,dt = 2t\cdot\frac{-1}{n\pi}\cos(n\pi t) - \int -\frac{2\cos(n\pi t)}{n\pi}\,dt = \frac{-2t}{n\pi}\cos(n\pi t) + \frac{2}{n^2\pi^2}\sin(n\pi t)$$

$$= \frac{2}{n^2\pi^2}(-\pi n t\cdot\cos(n\pi t) + \sin(n\pi t))$$

Substituting the integral with a definite integral,

$$b_n = \int_{-1}^{1} 2t\cdot\sin(n\pi t)\,dt = \frac{2}{n^2\pi^2}([-\pi n t\cdot\cos(n\pi t) + \sin(n\pi t)]_{-1}^{0} - [-\pi n t\cdot\cos(n\pi t) + \sin(n\pi t)]_{0}^{1}) + 0$$

$$= \frac{2}{n^2\pi^2}\{[(0+0) - (\pi n\cdot\cos(-\pi n) + \sin(-n\pi))] - [(-\pi n\cdot\cos(\pi n) + \sin(n\pi)) - (0+0)]\}$$

$$= \frac{2}{n^2\pi^2}(-\pi n\cdot\cos(\pi n) + \sin(\pi n) + \pi n\cdot\cos(\pi n) - \sin(\pi n)) = \frac{2}{n^2\pi^2}\cdot 0 = 0$$

Using the relation aforementioned, the Fourier coefficients c_n of the exponential Fourier series can also be calculated.

$$c_n = \frac{a_n}{2} - \frac{b_n}{2}i = \frac{\frac{4-4(-1)^n}{n^2\pi^2}}{2} - \frac{0}{2}i = \frac{2-2(-1)^n}{n^2\pi^2}$$

Therefore, the Fourier series of a periodic function $f(x) = \begin{cases} 2t+2, -1 < t \leq 0 \\ -2t+2, 0 < t \leq 1 \end{cases}, t \in \mathrm{R}$ with a period of 2 is $f(t) = 1 + \sum_{1}^{\infty}\frac{4-4(-1)^n}{n^2\pi^2}\cos(n\pi t)$, or $f(t) = \sum_{-\infty}^{\infty}\frac{2-2(-1)^n}{n^2\pi^2}e^{in\pi t}$ in its exponential form.

Calculating the Error of the Fourier Series Approximation

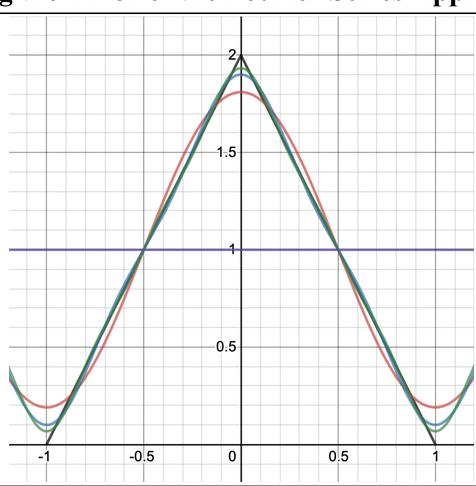


Fig. 5 Fourier Series Approximation of the Original Function

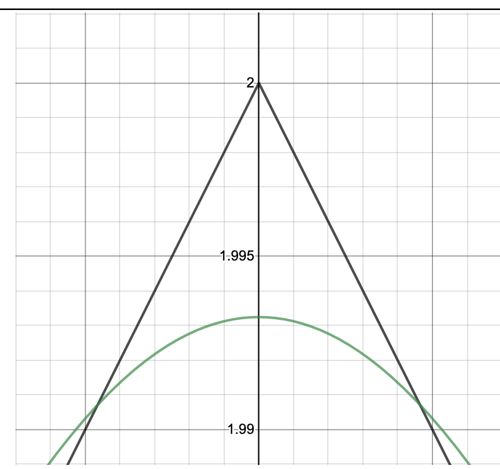


Fig. 6 Zoomed in Portion of Fourier Series Approximation at the Vertex

Fig. 5 shows the plot of this Fourier series along with its original function, where the purple line is the Fourier series with only the 0th term $\left(\frac{a_0}{2}\right)$, red with 1 term, blue 3, green 5, and black the original function. It should be noted that the Fourier series with even number of terms are not plotted because $4-4(-1)^n$ cancel to 0, which makes the Fourier series with $2m-1$ terms the same as that with $2m$ terms where $m \in N$. Therefore, the Fourier series with $n = 2m-1$ has m terms (not counting the constant term), and this convention will be used throughout this example. As shown in the graph, as the number of terms in the Fourier series increases, the Fourier series approximates the given function better.

In order to determine how well the Fourier series approximates the original function, the error of the two curves should be calculated. For a regression line, the root mean square error (RMSE) helps to measure how spread out a data is by calculating the square root of the sum of the squared error of each sample point from the regression line, where y_i represents the i[th] sample of the data and $\hat{y}_i$ represents its corresponding sample of the regression line (Holmes, 2000).

$$RMSE = \sqrt{\sum_{i=1}^{n} \frac{(\hat{y}_i - y_i)^2}{n}}$$

The Riemann sum, which represents an approximation of an integral with a finite sum, helps to approximate the area of a function by taking sample points of the graph with a fixed sample rate. A Riemann sum is defined as the following function, where S represents the sum, n the number of partitions, $\Delta x_i = x_i - x_{i-1}$, and $x_i^* \epsilon [x_{i-1}, x_i]$. x_i^* can be any point between x_{i-1} and x_i as the difference between x_i and x_{i-1} approaches 0 as Δx_i approaches 0 (Weisstein, n.d.).

$$S = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

As n approaches infinity and Δx_i approaches 0, the Riemann sum approaches the true sum of the function, which can also be written as the integral of the function from a to b.

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\, dx$$

For a function of n partitions, Δx_i equals to $\frac{1}{n}$ of the length of the defined domain, which would also equal to the period of a periodic function. The mean integrated squared error (MISE), or the error difference between two curves, of a function can then be calculated by replacing the summation in the RMSE formula with a definite integral, integrated from $-L$ to L for a function with a period of $2L$. The absolute value was included in the equation to help us calculate the difference between complex functions (Bevelacqua, n.d.).

$$\|\mathrm{f} - \mathrm{g}\| = \sqrt{\lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)^2 \Delta x_i} = \sqrt{\int_{-L}^{L} |f(x) - g(x)|^2\, dx}$$

The MISE of two functions has the same definition in the time domain. Let us do a calculation for the MISE for the Fourier series with one term ($n = 1$) (compared to the original function). Let $f(t) = \begin{cases} 2t + 2, -1 < t \le 0 \\ -2t + 2, 0 < t \le 1 \end{cases}$ and $g(t) = 1 + \sum_1^1 \frac{4-4(-1)^n}{n^2\pi^2} \cos(n\pi t)$, the MISE is then calculated as follows. Since both $f(t)$ and $g(t)$ are both even functions, the integration of the difference between $f(t)$ and $g(t)$ is also equal to two times the integral of either side of the function relative to the y-axis.

$$\|f(t) - g(t)\| = \sqrt{\int_{-1}^{1} \left| \begin{cases} 2t + 2, -1 < t \le 0 \\ -2t + 2, 0 < t \le 1 \end{cases} - \left(1 + \sum_{n=1}^{1} \frac{4 - 4(-1)^n}{n^2\pi^2} \cos(n\pi t)\right) \right|^2 dt}$$

$$= \sqrt{\int_{-1}^{0} \left|(2t + 2) - \left(1 + \frac{4 - 4(-1)}{\pi^2} \cos(\pi t)\right)\right|^2 dt + \int_{0}^{1} \left|(-2t + 2) - \left(1 + \frac{4 - 4(-1)}{\pi^2} \cos(\pi t)\right)\right|^2 dt}$$

$$= \sqrt{2\int_{-1}^{0} \left|2t + 1 - \frac{8}{\pi^2}\cos(\pi t)\right|^2 dt}$$

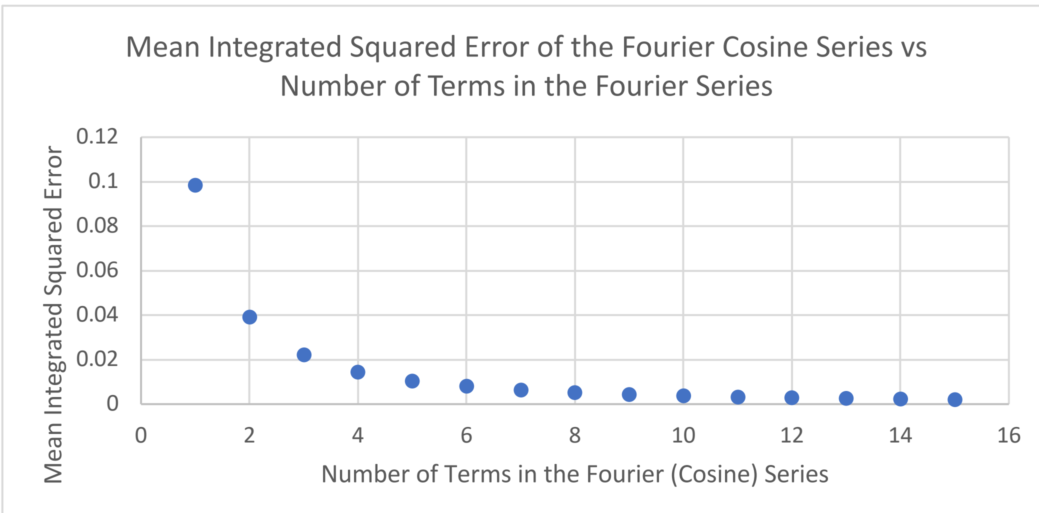
In order to calculate this integral, the intersections of the functions with the x-axis should be found. Since $2t + 1 - cos(\pi t) = 0$ is a non-linear function, the result is hard to determine using an analytical method. Using Wolfram Alpha, an online calculator, the roots that were calculated were $t \approx -0.874$, $t = -0.5$, and $t \approx -1.26$ (WolframAlpha, 2009).

Since $2(-1)+1-\frac{8}{\pi^2}\cos(-\pi) = -1+\frac{8}{\pi^2} = -0.189$ is negative, substituting the values into the integral gives:

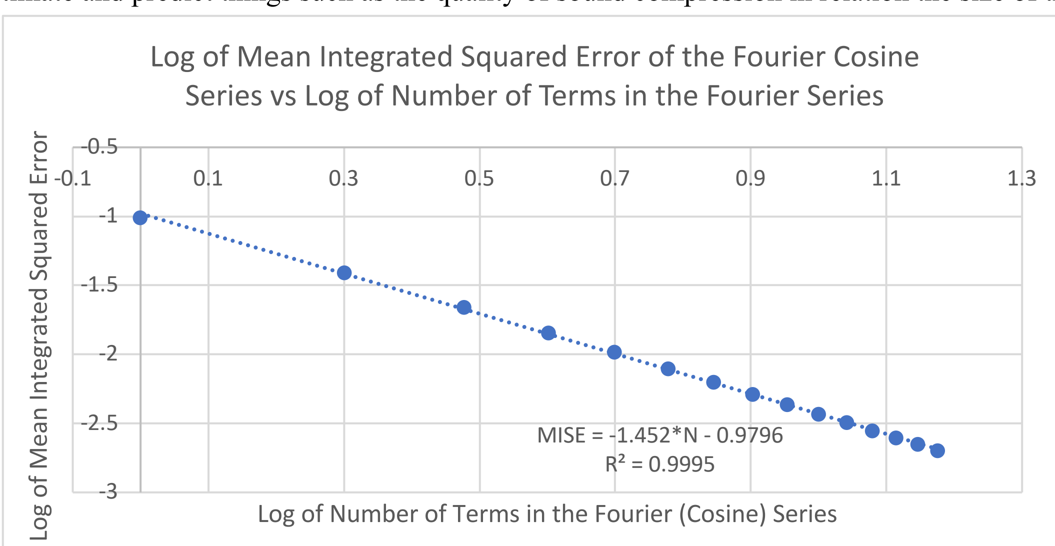
$$\sqrt{2\int_{-1}^{0}\left|2t+1-\frac{8}{\pi^2}\cos(\pi t)\right|^2 dt} = \sqrt{4\left(\int_{-1}^{-0.874} -2t-1+\frac{8}{\pi^2}\cos(\pi t)\,dt + \int_{-0.874}^{-0.5} 2t+1-\frac{8}{\pi^2}\cos(\pi t)\,dt\right)}$$

$$= 2\sqrt{\left(-t^2-t+\frac{8}{\pi^3}\sin(\pi t)\right)\Big|_{-1}^{-0.874} + \left(t^2+t-\frac{8}{\pi^3}\sin(\pi t)\right)\Big|_{-0.874}^{-0.5}} = 0.0982$$

A computer program was used to help calculate the MISE of the function with a larger number of different Fourier coefficients. Only 15 samples for the MISE (excluding the Fourier series with only the 0th term) were plotted for clarity of the graph shown below. As shown, the MISE decreases quickly at first and gradually approaches 0 as the number of terms in the Fourier cosine series increases, meaning that Fourier series can achieve compression of sound or images while preserving a high quality by replacing the function with a Fourier series that has a limited number of terms.



While plotting the graph, I discovered that when the data was plotted onto a log-log plot (as shown below), a linear trendline fitted the data points to a high degree, which means that the MISE of this function is in a power relationship with the number of terms in the Fourier cosine series; the R^2 value is "a statistical measure of how close the data are to the fitted regression line", and an R^2 value of 0.9995 signifies that there is a strong correlation (Minitab Blog Editor, 2013). However, I have not been able to understand why there is such a relation, so an extension of this exploration could be to investigate the rationale behind the power trend and determine whether this trend can be generalized to every function. If the trend can be generalized to other functions, then it can be applied to the field of signal processing to estimate and predict things such as the quality of sound compression in relation the size of the audio file.



In Fig. 9 and Fig. 10, three lines are plotted, wherein black represents the original function, red the Fourier cosine series with 1 and 3 terms respectively, and their respective absolute difference with the original function at each data point. Since the difference is always maximized when the function meets a vertex, it can be known that although a Fourier series does not approximate sharp edges (the vertex) well, it performs well on smooth edges (anywhere else on the function). As a result, compression methods such as JPEG for images can generally help to decrease the size of a file without being noticed because it eliminates a considerable amount of information that cannot be easily detected by the human eye, whereas PNG files are generally used by animators to preserve the details of the edges (Mathis, 2015).

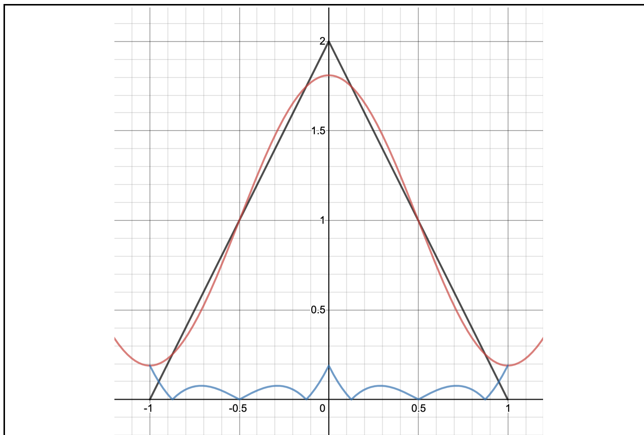


Fig. 9 Absolute Error of Fourier Series with 1 Term

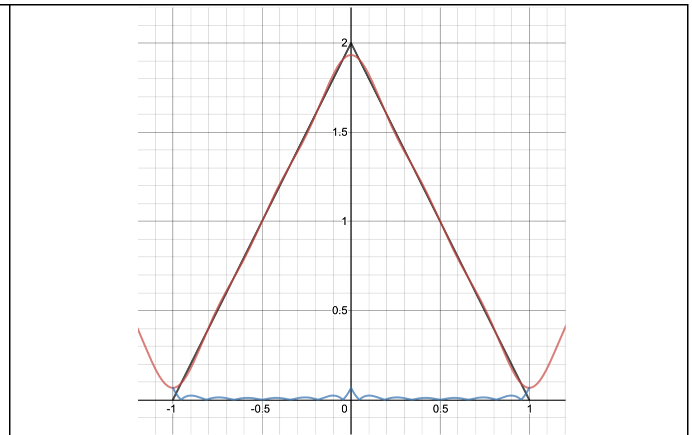


Fig. 10 Absolute Error of Fourier Series with 3 Terms

This example gives a perspective of some properties of the Fourier series, but it does not provide a rationale for why Fourier series are used since the function $f(t) = \begin{cases} 2t+2, -1 < x \leq 0 \\ -2t+2, 0 < x \leq 1 \end{cases}$ appears to be much simpler than the function $f(t) = 1 + \sum_1^1 \frac{4-4(-1)^n}{n^2\pi^2} \cos(n\pi t)$. It also does not achieve the aim to trace a figure with combinations of epicycles.

Plotting a Maple Leaf

To plot the maple leaf, a computer program was used to trace out the boundaries of the figure and take sample points due to the complexity of the figure. Since we now have discrete sample points as our input, a variant of the Fourier series, the discrete Fourier transform (DFT), was used. The DFT converts an input sequence into its frequency domain representation in discrete integer frequencies, and the terms that are calculated can then be used to plot the maple leaf figure. Although the proof is not shown, online sources do show that the DFT of n samples of a sampled signal is equal to n times the Fourier series coefficients of the same signal, meaning that the DFT of n samples produces the same output as the Fourier series for a continuous periodic function when divided by n (Smith, 2019). It should be noted that the maple leaf figure is periodic in the sense that it forms a closed loop when it's the base frequency traverses over a period. Although there exists the discrete-time Fourier transform (DTFT), which converts an input sequence into a continuous function of frequency, the DFT was used since discrete integer frequencies allows us to plot an image using circles with discrete radii and frequencies.

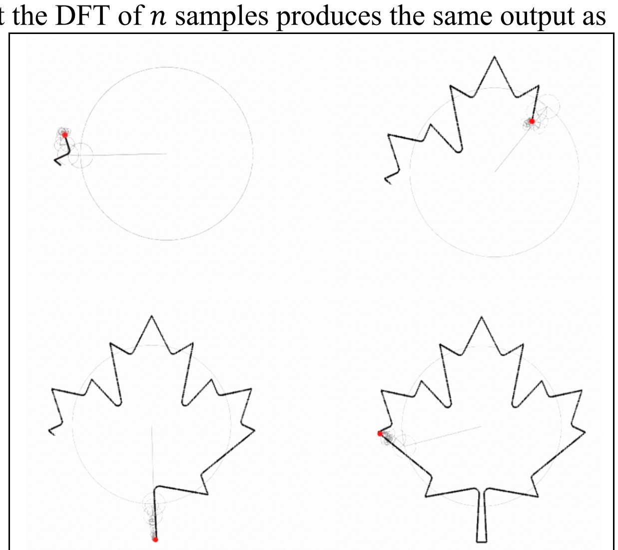


Fig. 11 Epicycle Representation of Maple Leaf

I wrote a computer program in MATLAB to trace out the maple leaf figure and perform the DFT, as shown in Fig. 11. In order to trace out the 2-dimensional figure using circles, the exponential form of the Fourier series with complex Fourier coefficients were used. The figure was also centered by setting the 0^{th} constant term in the Fourier series to 0.

Note that the commutative property of addition allows us to organize the terms in the Fourier series in any order. For a clear visual representation, the terms in the Fourier series were sorted in terms of decreasing amplitude, which is shown on the graph as circles with decreasing radius. This example shows how the Fourier series achieves compression visually; by only retaining the terms with a relatively large amplitude (large radius), a similar graph that may be visually indistinguishable can still be drawn.

While I was trying to plot the maple leaf, I also thought that it would be interesting to calculate the perimeter of the figure that is traced out with the combination of circles. Through previous learning, I have learned that the arc length of a parametric curve given by $x = f(t)$ and $y = g(t)$, where $f(t)$ and $g(t)$ represent two functions with the parameter t, can be calculated. Since complex numbers have both an x (real) and y (imaginary) component, I thought that the complex number series can also be used to calculate its arc length in a similar manner.

Given that we have a parametric curve given by $x = f(t)$ and $y = g(t)$, with $a \leq t \leq b$. Let us assume that there is an infinitesimal curve ds. Using Pythagoras's theorem, it can be seen that such a curve can be expressed in terms of dx and dy, infinitesimal sections of the x and y axis through the following relation.

$$ds^2 = dx^2 + dy^2$$

$$\therefore ds = \sqrt{dx^2 + dy^2}$$

Using integration and the chain rule, the arc length of a parametric curve can be calculated as follows (Dawkins, 2018).

$$L = \int ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt$$

A complex number allows the real and imaginary component of a function to be calculated separately, where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ correspond to the derivatives of the real and imaginary components. Since the derivative of $f(t)$ calculates both the derivative of the real and imaginary components of the function, and the absolute value (or modulus) of a complex number and can also be calculated by taking the square root of the sum of both the square of its real and imaginary components, the arc length of $f(t) = \sum_{-\infty}^{\infty} c_n e^{in\omega t}$ can be calculated as $L = \int_a^b |f'(t)| dt$.

A simple example can be used to verify this result. Let us take a system with two rotating circles, one with a radius of 4 and a counterclockwise rotation with an angular frequency of 1 rotation per second, and the other with a radius of 1 and a counterclockwise rotation with an angular frequency of 4 rotations per second (Fig. 12). As the epicycle (smaller circle) traces out its path, it traces out an epicycloid, a path traced out by a point on the circumference of a circle rolled on the outside of a fixed circle without slipping (Weisstein, Epicycloid., 2004). Therefore, this system can also be viewed as a cycloid, a point on the rim of a circle rolled along a straight line, that is curved around a circle (Fig. 13) (Cycloid, 2008). Note that only three curves are shown in the graph since the cycloid that is traced out is not on a flat line.

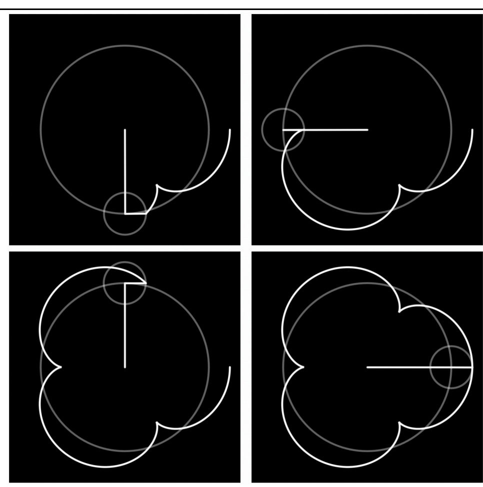


Fig. 12 A system with an epicycle of radius 1 and deferent of radius 4

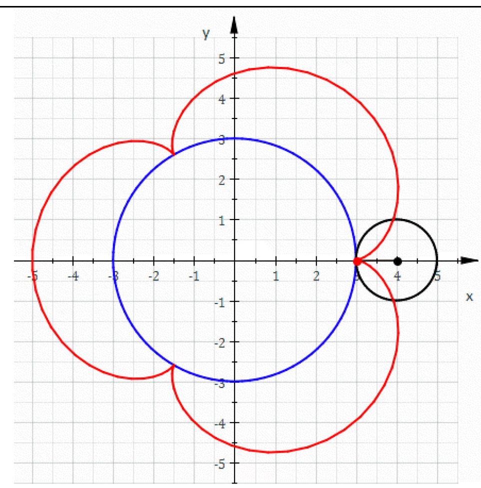


Fig. 13 A circle with a radius of 1 rolled around a circle with a radius of 3

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It is known that a cycloid can be expressed in terms of a parametric equation with $x = r(t - \sin(t))$ and $y = r(1 - \cos(t))$ (Gilbert & Schmidt, 2017). The arc length of the cycloid S can then be calculated using the formula to calculate the arc length of a parametric equation.

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{r^2(1-\cos(t))^2 + r^2(-\sin(t))^2}\, dt$$

$$= r\int_0^{2\pi} \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}\, dt = r\int_0^{2\pi} \sqrt{2}\sqrt{1-\cos(t)}\, dt = 2r\int_0^{2\pi} \sqrt{\sin^2\left(\frac{t}{2}\right)}\, dt$$

$$= 2r\int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt = 2r\left(-2\cos\left(\frac{t}{2}\right)\right)\Big|_0^{2\pi} = -4r(-1) - \left(-4r(1)\right) = 8r$$

As shown above, the arc length of a cycloid is equal to eight times its radius, so the total length of the cycloid that is traced out from an epicycle of radius 1 that rotates 4 times is $8 \cdot 1 \cdot 4 = 32$. The arc length of the epicycle system is calculated as follows.

$$S = \int_0^1 \left|\frac{d}{dt}\left(4e^{i2\pi t} + e^{4\cdot(i2\pi t)}\right)\right| dt = 8i\pi\int_0^1 \left|e^{i2\pi t} + e^{i8\pi t}\right| dt$$

$$= 8\pi\int_0^1 \left| -(\sin(2\pi t) + \sin(8\pi t)) + i(\cos(2\pi t) + \cos(8\pi t))\right| dt$$

$$= 8\pi\int_0^1 \sqrt{(-(\sin(2\pi t) + \sin(8\pi t)))^2 + (\cos(2\pi t) + \cos(8\pi t))^2}\, dt$$

$$= 8\pi\int_0^1 \sqrt{\sin^2(2\pi t) - 2\sin(2\pi t)\sin(8\pi t) + \sin^2(8\pi t) + \cos^2(2\pi t) + 2\cos(2\pi t)\cos(8\pi t) + \cos^2(8\pi t)}\, dt$$

$$= 8\pi\int_0^1 \sqrt{2 - 2\sin(2\pi t)\sin(8\pi t) + 2\cos(2\pi t)\cos(8\pi t)}\, dt = 8\pi\int_0^1 \sqrt{2 + 2\cos(2\pi t - 8\pi t)}\, dt$$

$$= 8\sqrt{2}\pi\int_0^1 \sqrt{1 + \cos(2\pi t)}\, dt = 8\sqrt{2}\pi\int_0^1 \sqrt{1 + \cos(2\pi t)}\, dt = 8\sqrt{2}\pi\int_0^1 \sqrt{2\cos^2(\pi t)}\, dt = 16\pi\int_0^1 |\cos(\pi t)| dt$$

$$= 16\pi\left(\int_0^{0.5} \cos(\pi t)\, dt - \int_{0.5}^1 \cos(\pi t)\, dt\right) = \frac{16\pi}{\pi}\left(\sin(\pi t)\,\Big|_0^{0.5} - \sin(\pi t)\,\Big|_{0.5}^1\right) = 16((1-0) - (0-(-1))) = 32$$

Both calculations produced the same arc length, so this method is verified. Using Wolfram Alpha, I have also checked that the result of this definite integral is correct (WolframAlpha, 2009). With this result, we can also calculate the arc length of each curve traced out in the system, which is $32 \div 3 = \frac{32}{3}$.

To calculate the perimeter of the maple leaf Fourier series, I tried doing the computation by plugging in all the terms calculated using the DFT. However, the queries didn't work well since there were so many terms, so I wrote a program using WolframScript to calculate the perimeter of the maple leaf. The perimeter of the maple leaf was calculated to be $5404.17 \approx 5400$. Although this result does not have any units, for a reference, the largest circle in Fig. 11 had a radius of $939.34 \approx 900$ and a circumference of $5902.06 \approx 5900$. This result means that maple leaf has a perimeter with a ratio of $\frac{5404.17}{5902.06} \approx 91.6\%$ relative to the largest circle in its Fourier series. As aforementioned, the more terms that are calculated in the Fourier series, the better the Fourier series represents the original function. Such a result provides a perspective that relatively fewer terms in the Fourier series may be used to plot the maple leaf figure (while preserving its outline) for ease of computation (including the calculation for its perimeter) and more efficient storage of data.

Conclusion and Extensions

I started out seeking to understand the Fourier series through drawing figures with circles. Throughout the exploration, I have discovered many properties of the Fourier series, including an increasing similarity compared to the original function as the number of terms in the Fourier series increases and the fact that Fourier series do not approximate edges well. Both of these properties show why applications of Fourier analysis, such as image and audio compression, are used for data compression when much information can be excluded.

In the exploration, I have also learned about the two forms of the Fourier series – trigonometric and exponential. The exponential Fourier series allows complex and parametric functions to be plotted. With many properties similar to those of parametric equations, the exponential Fourier series can also be used to estimate the length (perimeter) of the function, as well as the area of the function. In addition, the exponential Fourier series provides a way to apply a phase shift to each Fourier coefficient and a way to visualize the series using the concept of deferent and epicycles. Therefore, the exponential Fourier series is also a more compact representation of the Fourier series.

As an extension, I would like to learn about the Fourier transform, since Fourier series have the limitation that a function must be periodic and infinitely repeating throughout the defined domain. Since most signals in real life are not periodic, applications including image compression, filtering, and analysis require the use of the Fourier transform (Cheever, n.d.). In addition, the wavelet transform also decomposes functions into different frequency components, and they provide much better compared to the Fourier transform when there are discontinuities are sharp spikes present in a signal, although both methods require a more rigorous mathematical background for further study (Graps). Such investigations may help us answer the question of how visual and auditory signals could be further compressed while preserving their quality, and they may have a large global impact, especially in the current technological era.

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